

Cosmological dynamics of exponential gravity

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Abstract. We present a detailed investigation of the cosmological dynamics based on $\exp(-R/\Lambda)$ gravity. We apply the dynamical system approach to both the vacuum and matter cases and obtain exact solutions and their stability in the finite and asymptotic regimes. The results show that cosmic histories exist which admit a double de-Sitter phase which could be useful for describing the early and the late-time accelerating universe.

1. Introduction

In recent years several observational surveys appear to show that the universe is currently undergoing an accelerated expansion phase (for a review see [1]). On the other hand, astrophysical observations on galactic scales gave us a clear indication that the amount of the luminous matter is not enough to account for the rotation curves of galaxies. These remarkable observations have radically changed our ideas about the evolution of the universe. In fact, the only way in which this behaviour can be explained within the standard Friedmann general relativistic cosmology framework is to invoke two dark matter components, one with a conventional dust equation of state ($w = 0$) needed to fit astrophysical data on galactic scales (Dark Matter) and the other with a more exotic equation of state ($w < -\frac{1}{3}$) needed to explain the current accelerated expansion phase of the universe (Dark Energy). There is little doubt that unraveling the nature of Dark matter and Dark Energy is currently one of the most important problems in theoretical physics.

One approach to the problem of Dark Energy that has received considerable attention in the last few years is the modification of General Relativity on cosmological scales. In this approach one supposes that Dark energy is a manifestation of a non-Einsteinian behaviour of the gravitational interaction rather than a new form of energy density. The introduction of corrections to the Hilbert Einstein action and their effects have been studied for long time and are believed to be unavoidable when the quantum nature of the universe is introduced in General Relativity (GR).

In the last few years many different extended versions of the Einstein theory of gravity have been proposed. One of the most studied approaches is higher order theories of gravity [13 – 42], in which the gravitational action is nonlinear in the Ricci curvature and/or its derivatives [6, 7, 8]. These theories have a number of interesting features on cosmological and astrophysical scales. In fact they are known to admit natural inflation phases [4] and to explain the flattening of the galactic rotation curves [9]. Another very interesting feature of these models is that the higher

order corrections to Hilbert-Einstein action can be viewed as an effective fluid which can mimic the properties of Dark Energy [5].

One of the the main problems that occurs in the study of higher order theories of gravity is that finding exact cosmological solutions is extremely difficult due to the high degree of non-linearity exhibited by these theories. This problem can be partially addressed using a suitable choice of generalized coordinates in which the field equations can be written as a system of first-order autonomous differential equations together with a constraint equation [10]. In this way we can exploit the methods of dynamical systems theory [2] in order to both understand the qualitative behavior of the cosmological dynamics and obtain special exact solutions of the cosmological equations. The general approach allowing one to analyze higher order gravity with dynamical systems techniques has been presented elsewhere [3].

In this paper we will apply this approach to an important class of theories with Lagrangian density

$$\mathcal{A} = \int d^4x \mathcal{L} = \int d^4x \sqrt{-g} \left[e^{-\frac{R}{\Lambda}} + L_M \right], \quad (1)$$

where Λ is the cosmological constant. This Lagrangian has some interesting features, first of all its treatment is equivalent to one involving a combination of powers of the Ricci scalar and, secondly it reduces to

$$\exp\left(-\frac{R}{\Lambda}\right) = 1 - \frac{R}{\Lambda} + O[R^2], \quad (2)$$

In the small curvature limit, which is equivalent to Hilbert-Einstein action.

This paper has been arranged as follows. In section 2, we present the basic equations of the model. In section 3 and 4 we find exact solutions and their stability in the vacuum and matter cases respectively. Finally in section 5 we present a discussion of the results and present our conclusions.

In what follows we will use natural units ($\hbar = c = k_B = 8\pi G = 1$) and the signature $(+, -, -, -)$.

2. Basic Equations

The general action for a fourth order theory of gravity is:

$$\mathcal{A} = \int d^4x \sqrt{-g} [f(R) + L_M], \quad (3)$$

where $f(R)$ is a function of Ricci scalar R and L_M is the standard matter Lagrangian density. Varying this action with respect to the metric gives

$$G_{\mu\nu} = T_{\mu\nu}^{ToT} = T_{\mu\nu}^M + T_{\mu\nu}^R = \frac{1}{f'(R)} \tilde{T}_{\mu\nu}^M + \frac{1}{f'(R)} \left(\frac{1}{2} g_{\mu\nu} [f(R) - Rf'(R)] + f'(R)^{;\alpha\beta} (g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\beta} g_{\mu\nu}) \right), \quad (4)$$

where the prime denotes the derivative with respect to R , $T_{\mu\nu}^M$ is the effective stress-energy tensor for standard matter, which is assumed to be a perfect fluid and $T_{\mu\nu}^R$ is the stress-energy tensor of the curvature *fluid* which represents an additional source term of purely geometrical origin [11]. By assuming $f(R) = \exp(-R/\Lambda)$ we obtain

the field equations:

$$G_{\mu\nu} = -\Lambda e^{R/\Lambda} \tilde{T}_{\mu\nu}^M \quad (5)$$

$$- e^{-R/\Lambda} \left[\frac{1}{2} g_{\mu\nu} (\Lambda + R) + \frac{1}{\Lambda^2} (R^{;\alpha} R^{;\beta} + \Lambda R^{;\alpha\beta}) (g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\beta} g_{\mu\nu}) \right]. \quad (6)$$

In the case of the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric, the above equations reduce to:

$$\begin{aligned} H^2 + \frac{k}{a^2} - \frac{H}{\Lambda} \dot{R} + \frac{R}{6} + \left(\frac{\Lambda}{6} + \frac{\Lambda \rho}{3e^{-R/\Lambda}} \right) &= 0, \\ 2\frac{\ddot{a}}{a} + \frac{R}{3} - \frac{H}{\Lambda} \dot{R} + \frac{1}{\Lambda^2} \dot{R}^2 - \frac{1}{\Lambda} \ddot{R} - \frac{\Lambda \rho}{3e^{-R/\Lambda}} (1 + 3w) + \frac{\Lambda}{3} &= 0, \end{aligned} \quad (7)$$

with,

$$R = -6 \left(\frac{\ddot{a}}{a} + H^2 + \frac{k}{a^2} \right), \quad (8)$$

where $H = \frac{\dot{a}}{a}$ is the usual Hubble parameter and k is the spatial curvature. The Bianchi identities applied to the total stress-energy tensor $T_{\mu\nu}^{TOT}$ lead to the energy conservation equation for standard matter [2]:

$$\dot{\rho} + 3H\rho(1 + w) = 0. \quad (9)$$

3. The vacuum case

In the vacuum case ($\rho = 0$) equation (7) and (8) can be written as a closed system of first order differential equations using the dimensionless variables,

$$x = \frac{\dot{R}}{\Lambda H}, \quad y = \frac{R}{6H^2}, \quad z = \frac{\Lambda}{6H^2}, \quad K = \frac{k}{a^2 H^2}. \quad (10)$$

It is clear that the variables y and z are a measure of the expansion normalized Ricci curvature and the cosmological constant respectively, K is the spatial curvature parameter of the Friedmann model, while x is a measure of the time rate of change of Ricci curvature. The evolution equations for the variables (10) are given by

$$\begin{aligned} x' &= 2z + 2K - 2 + x(1 + x + y + K), \\ y' &= xz + 2y(2 + y + K), \\ z' &= 2z(2 + y + K), \\ K' &= 2K(y + 1 + K), \end{aligned} \quad (11)$$

where the prime represents the derivatives with respect to the time variable $\mathcal{N} = \ln a$. This system is completed with the Friedmann constraint,

$$1 + K + y + z - x = 0, \quad (12)$$

which defines a hyperplane in the total phase space of the system. Consequently, all solutions of the dynamical system will be located in a non-compact submanifold of the phase space associated with (11). The time derivative of (12) is nothing other than the Raychaudhuri equation.

3.1. Finite analysis

The dimensionality of the state space of the system (11) can be reduced by eliminating any one of the four variables using the constraint equation (12). If we choose to eliminate the x the dynamical equations become

$$\begin{aligned} y' &= y(4 + 2K + 2y + z) + z(1 + K + z), \\ z' &= 2z(2 + K + y), \\ K' &= 2K(1 + K + y), \end{aligned} \quad (13)$$

and the number of invariant submanifolds is maximized, making the analysis much easier. It is clear from the above equations that $z = 0$ (corresponding to the $(K - y)$ plane) is an invariant submanifold. Specifically, If we choose $z \neq 0$ as an initial condition for our cosmological evolution, any general orbit can only approach $z = 0$ asymptotically. This implies that no orbit crosses the $(K - y)$ plane and consequently no global attractor can exist.

Setting $K' = 0$, $y' = 0$, $z' = 0$, we obtain four fixed points. We can obtain exact cosmological solutions at these points using the equation,

$$\dot{H} = -(y + K + 2)H^2. \quad (14)$$

In fact, at any fixed point, equation (14) reduces to

$$\dot{H} = -\frac{1}{\alpha}H^2, \quad (15)$$

where

$$\alpha = (y_* + K_* + 2)^{-1}, \quad \alpha \neq 0 \quad (16)$$

and the quantities X_* are meant to be calculated at the fixed point. Equation (14) applies to both the matter and vacuum cases and describes a general power law evolution of the scale factor. In addition integrating with respect to time we obtain

$$a = a_o(t - t_o)^\alpha. \quad (17)$$

This means that by finding the value of α at a given fixed point, we can obtain the solutions associated with it using equation (14).

In this way, points \mathcal{A}_v and \mathcal{B}_v (see Table 1) are found to represent Milne and power-law evolutions respectively. However, by direct substitution into the cosmological equations it can be shown that these fixed points cannot be considered as physical points, which means that although we can choose initial conditions as close as we want to these points, the cosmology will never evolve according to these solutions.

For the points \mathcal{C}_v and \mathcal{D}_v we have $y_* + K_* + 2 = 0$. In this case

$$\dot{H} = 0, \quad (18)$$

which implies

$$a = a_o e^{\gamma(t-t_o)}. \quad (19)$$

The value of the constant γ can be obtained by direct substitution into equations (7) and (8). For both \mathcal{C}_v and \mathcal{D}_v we obtain

$$\gamma = \pm \sqrt{\frac{\Lambda}{6}}, \quad (20)$$

so they represent an exponential evolution. The contracting or expanding nature of this solution depends on the direction of approach of the orbits with respect to

the hypersurface $y + K + 2 = 0$. This hypersurface divides the phase space in two hypervolumes characterized by a contracting or expanding evolution. In particular, for $y < -K - 2$ the orbits describe a contracting universe, while for $y > -K - 2$ they represent an expanding one.

The stability of the hyperbolic fixed points \mathcal{A}_v , \mathcal{B}_v and \mathcal{D}_v is obtained by using Hartman-Grobman theorem, while for the point \mathcal{C}_v , which is non-hyperbolic, we use the local center manifold theorem to find it's stability. A brief review of this approach is presented in the Appendix.

In our case, using the transformation

$$\begin{aligned} y &= u - 2v - m, \\ z &= 4m, \\ k &= v \end{aligned} \tag{21}$$

the system (13) can be written in the diagonal form

$$\dot{u} = -4u + 12m^2 + 2mu + 2u^2 - 4mv - 2uv, \tag{22}$$

$$\dot{v} = -2v - 2mv + 2uv - 2v^2, \tag{23}$$

$$\dot{m} = -2m^2 + 2mu - 2mv \tag{24}$$

near the non-hyperbolic fixed point \mathcal{C}_v . By substituting the expansions

$$h1(m) = am^2 + bm^3 + O(m^4), \tag{25}$$

$$h2(m) = cm^2 + dm^3 + O(m^4) \tag{26}$$

into equations (A.4) and (A.6) and then solving for a , b , c and d , we obtain

$$h1(m) = 3m^2 + \frac{9}{2}m^3 + O(m^4), \quad h2(m) = O(m^4). \tag{27}$$

Substituting this result into equation (A.4) then yields

$$\dot{m} = -2m^2 + O(m^3) \tag{28}$$

on the center manifold $W^c(\mathbf{0})$, near the point \mathcal{C}_v . This implies that the point \mathcal{C}_v is a saddle-node i.e. it behaves like a saddle or an attractor depending on the direction from which the orbit approaches. The local phase portrait in the neighborhood of \mathcal{C}_v is shown in Figure 1.

If one considers now the transformation (21) one realizes that $m \propto z$, so that \mathcal{C}_v is an attractor for $z > 0$ and a saddle for $z < 0$. This is also clear from Figure 2 in which the invariant submanifold $z - y$ is depicted.

Finally, it is useful to derive an expression for the deceleration parameter q in terms of the dynamical variables:

$$q = -\frac{\dot{H}}{H^2} - 1 = -(y + K + 1). \tag{29}$$

It follows that $q > 0 \Rightarrow (y + K + 1) < 0$. This condition is satisfied only for the point \mathcal{C}_v as expected by looking at the solution associated with this fixed point. In Figure 3 we give the location of the $q = 0$ plane relative to the fixed points \mathcal{A}_v , \mathcal{C}_v and \mathcal{B}_v .

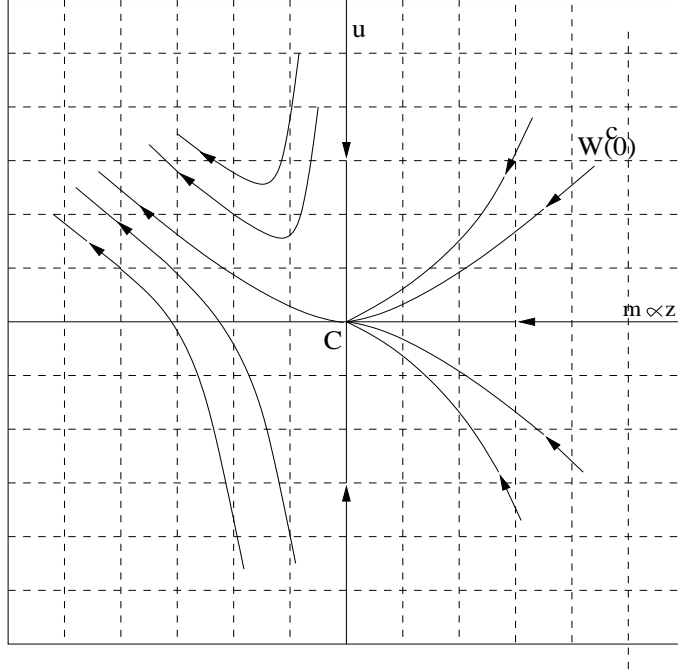
3.2. Asymptotic analysis

In this section we will determine the fixed points at infinity and study their stability. To simplify the asymptotic analysis we need to compactify the phase space using the Poincaré method. Transforming to polar coordinates $(r(N), \theta(N), \phi(N))$:

$$z \rightarrow r \cos \theta, \quad K \rightarrow r \sin \theta \cos \phi, \quad y \rightarrow r \sin \theta \sin \phi$$

Table 1. Coordinates of the fixed points, eigenvalues, stability and solutions for $\exp(-\frac{R}{\Lambda})$ -gravity in vacuum.

Point	Coordinates(y,z,K)	Eigenvalues	Stability	Solution
\mathcal{A}_v	[0, 0, 0]	[2, 4, 4]	repeller	$a = a_o(t - t_o)$
\mathcal{B}_v	[0, 0, -1]	[-2, 2, 2]	Saddle	$a = a_o(t - t_o)^{\frac{1}{2}}$
\mathcal{C}_v	[-2, 0, 0]	[-4, -2, 0]	Saddle-node	$a = a_o e^{\gamma(t-t_o)}$
\mathcal{D}_v	[-2, 1, 0]	$[-\frac{(3+\sqrt{17})}{2}, -2, \frac{(3+\sqrt{17})}{2}]$	Saddle	$a = a_o e^{\gamma(t-t_o)}$

**Figure 1.** The phase portrait for the system (12) in the neighbourhood of the fixed point C for $\exp(-R/\Lambda)$ -gravity in vacuum.

and substituting $r \rightarrow \frac{R}{1-R}$, the regime $r \rightarrow \infty$ corresponds to $R \rightarrow 1$. Using this coordinate transformation and taking the limit $R \rightarrow \infty$, the system (13) can be written as

$$R' \rightarrow \frac{1}{4} \left(8 \cos(\phi) \sin(\theta)^3 - \sin(\phi) (-7 \sin(\theta) + \sin(3\theta)) + 8 \cos(\theta)^2 \sin(\theta) (\chi) + 4 \cos(\theta) \sin(\theta)^2 \sin(\phi) (\chi) \right), \quad (30)$$

$$R\theta' \rightarrow \frac{-\cos(\theta)^2 \sin(\phi) (\cos(\theta) + \sin(\theta) (\chi))}{R-1}, \quad (31)$$

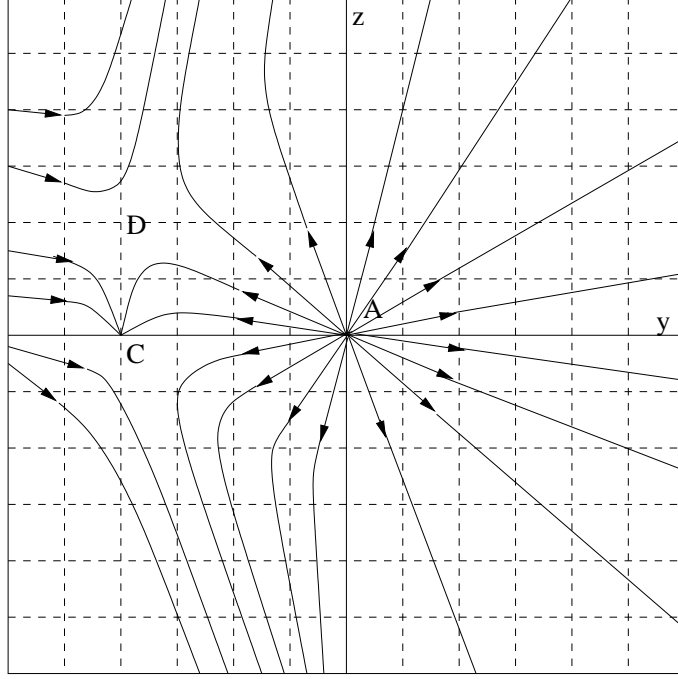


Figure 2. The phase space of the invariant submanifold $z = y$ for $\exp(-R/\Lambda)$ -gravity in vacuum.

$$R\phi' \rightarrow \frac{\cos(\phi) \cot(\theta) (\chi + \cot(\theta))}{R-1}, \quad (32)$$

where $\chi = \cos(\phi) + \sin(\phi)$. Since equation (30) does not depend on the coordinate R , we can find the fixed points of the above system using equations (31) and (32) only. The results are shown in Table 2.

Let us now derive the solution for the first fixed point, which corresponds to $K \rightarrow -\infty, z \rightarrow \infty$. In these limits the first equation of the system (13) reduces to

$$\dot{K} = 2K \quad (33)$$

and the equation for \dot{H} becomes

$$\dot{H} = KH^2. \quad (34)$$

Integrating equation (33) we obtain

$$K = -\frac{1}{2(N - N_\infty)}. \quad (35)$$

Substituting K back into equation (34) and solving for $(N - N_\infty)$, we obtain

$$(N - N_\infty) = [c_1 \pm \frac{3}{2}c_o(t - t_o)]^{\frac{2}{3}}. \quad (36)$$

The same procedure can be used to obtain solutions for the other asymptotic points. On the fixed circle \mathcal{I}_∞ we have three possible solutions corresponding to whether the sign of K and y are the same or different.

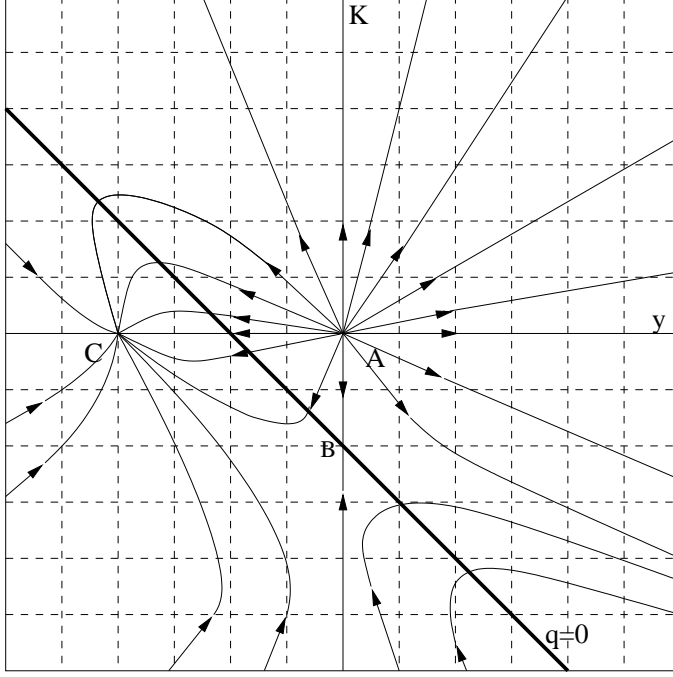


Figure 3. The location of the $q = 0$ plane relative to the fixed points \mathcal{A}_v , \mathcal{C}_v and \mathcal{B}_v for $\exp(-R/\Lambda)$ -gravity in vacuum.

The stability of the fixed points in Table 2 are found by expanding equations (31) and (32) up to second order and then applying the center manifold theorem to the resulting system (see Table 3).

For the fixed circle \mathcal{I}_∞ , the stability depends on the value of the angle ϕ :

$-\pi/4 < \phi < \pi/2$	Stable,
$\pi/2 < \phi < 3\pi/4$	Unstable,
$3\pi/4 < \phi < 3\pi/2$	Stable,
$3\pi/2 < \phi < 7\pi/4$	Unstable.

In the next section we will see how the introduction of matter modifies the picture we obtained in the vacuum case.

4. The matter case

In this case we can use the same dynamical variables we used for the vacuum case together with one additional variable D , that is related to the matter energy density:

$$x = \frac{\dot{R}}{\Lambda H}, \quad y = \frac{R}{6H^2}, \quad z = \frac{\Lambda}{6H^2}, \quad K = \frac{k}{a^2 H^2}, \quad D = \frac{\Lambda \rho}{3H^2 e^{R/\Lambda}}. \quad (37)$$

The definition of the variables reveals that not all of the phase space corresponds to physical situations. This becomes clear if we divide D by z . We obtain

$$\frac{D}{z} = 2\rho \exp\left(-\frac{R}{\Lambda}\right), \quad (38)$$

Table 2. Coordinates, eigenvalues and the Stability of the fixed points in the asymptotic regime for $\exp(-\frac{R}{\Lambda})$ gravity in vacuum.

Point	(θ, ϕ)	Eigenvalues	Coordinates	Solution
$\mathcal{A}_v \infty$	$[-\frac{\pi}{4}, 0]$	$[2, 0]$	$ K , z \rightarrow \infty \ y \rightarrow 0$	$(N - N_\infty) = [c_1 \pm \frac{3c_\phi}{2}(t - t_o)]^{\frac{2}{3}}$
$\mathcal{B}_v \infty$	$[\frac{\pi}{4}, -\pi]$	$[2, 0]$	$ K , z \rightarrow \infty \ y \rightarrow 0$	$(N - N_\infty) = [c_1 \pm \frac{3c_\phi}{2}(t - t_o)]^{\frac{2}{3}}$
$\mathcal{E}_v \infty$	$[-\frac{3\pi}{4}, -\frac{\pi}{2}]$	$[\frac{1}{2\sqrt{2}}, 0]$	$ y , z \rightarrow \infty \ K \rightarrow 0$	$(N - N_\infty) = [c_1 \pm \frac{c_\phi}{2}(t - t_o)]^2$
$\mathcal{F}_v \infty$	$[\frac{\pi}{4}, -\frac{\pi}{2}]$	$[\frac{1}{2\sqrt{2}}, 0]$	$ y , z \rightarrow \infty \ K \rightarrow 0$	$(N - N_\infty) = [c_1 \pm \frac{3c_\phi}{2}(t - t_o)]^{\frac{2}{3}}$
$\mathcal{G}_v \infty$	$[-\frac{\pi}{4}, \frac{\pi}{2}]$	$[\frac{1}{2\sqrt{2}}, 0]$	$ y , z \rightarrow \infty \ K \rightarrow 0$	$(N - N_\infty) = [c_1 \pm \frac{3c_\phi}{2}(t - t_o)]^{\frac{2}{3}}$
$\mathcal{H}_v \infty$	$[\frac{3\pi}{4}, \frac{\pi}{2}]$	$[\frac{1}{2\sqrt{2}}, 0]$	$ y , z \rightarrow \infty \ z \rightarrow 0$	$(N - N_\infty) = [c_1 \pm \frac{c_\phi}{2}(t - t_o)]^2$
$\mathcal{I}_v \infty$	$[\frac{\pi}{2}, \phi]$	$[0, -\chi \cos(\phi)]$	$y \rightarrow \pm\infty \ K \rightarrow \mp\infty$ $y \rightarrow +\infty \ K \rightarrow +\infty$ $y \rightarrow -\infty \ K \rightarrow -\infty$	$a = a_0 \exp[\gamma(t - t_0)]$ $(N - N_\infty) = [c_1 \pm \frac{c_\phi}{2}(t - t_o)]^2$ $(N - N_\infty) = [c_1 \pm \frac{3c_\phi}{2}(t - t_o)]^{\frac{2}{3}}$

Table 3. Coordinates of the asymptotic fixed points of $\exp(-R/\Lambda)$ -gravity in vacuum and their stability.

Point	Coordinates(y,z,K)	Stability
\mathcal{A}_∞	$[-\pi/4, 0]$	Unstable
\mathcal{B}_∞	$[\pi/4, -\pi]$	Stable
\mathcal{E}_∞	$[-3\pi/4, -\pi/2]$	Stable
\mathcal{F}_∞	$[\pi/4, -\pi/2]$	Stable
\mathcal{G}_∞	$[-\pi/4, \pi/2]$	Stable
\mathcal{H}_∞	$[3\pi/4, \pi/2]$	Stable

which has the same sign of ρ . This means that the sectors in the phase space for which the sign of D is different from the sign of z contain orbits in which standard matter violates the weak energy condition $\rho > 0$, and have to be discarded as not physical. As we will see this affect the sets of possible orbits for this model.

Following the same procedure we used in the vacuum case, we obtain an autonomous system equivalent to the cosmological equations with non-zero matter density:

$$\begin{aligned}
x' &= 2 + 2z + 2K + x(1 - x + y + K) - D(1 + 3w) , \\
y' &= xz + 2y(2 + y + K) , \\
z' &= 2z(2 + y + K) , \\
K' &= 2K(y + 1 + K) , \\
D' &= D(1 - 3w + 2y + 2K - x) ,
\end{aligned} \tag{39}$$

together with the constraint equation

$$1 + K + x + y - z - D = 0 , \tag{40}$$

where the prime again denotes the derivative with respect to the logarithmic time variable \mathcal{N} .

4.1. Finite analysis

The system (39) can be further simplified, by eliminating x using the constraint equation (40):

$$\begin{aligned} y' &= y(4 + 2K + 2y + z) + z(1 + K + D + z) , \\ K' &= 2K(1 + K + y) , \\ z' &= 2z(2 + y + K) , \\ D' &= D(2 - 3w + 3K + D + 3y + z) . \end{aligned} \quad (41)$$

We have three invariant submanifolds: $K = 0$, $z = 0$ and $D = 0$, so in this case also no global attractor can exist. Setting $K' = 0$, $y' = 0$, $z' = 0$ and $D' = 0$ we obtain seven fixed points.

As in the vacuum case, we can use the coordinates of these fixed points and equation (14) to find the behaviour of the scale factor at these points. In addition, the behaviour of the energy density ρ can be obtained from equation (9), which at a fixed point reads

$$\frac{\dot{\rho}}{\rho} = -3(1 + w)\frac{\alpha}{t} , \quad (42)$$

where α is defined by (15). However, direct substitution in the cosmological equations reveals that all the fixed points correspond to vacuum states.

Points \mathcal{A}_m and \mathcal{B}_m are found to represent Milne solutions while \mathcal{C}_m and \mathcal{D}_m represent a power law evolution. For points \mathcal{E}_m , \mathcal{F}_m and \mathcal{G}_m we find that $\dot{H} = 0$, which means that these points represent Einstein-de Sitter solutions. The exact solutions at these fixed points are summarized in Table 4.

As in the vacuum case, we use the Hartman-Grobman theorem together with the center manifold theorem to analyze the stability of all the fixed points. The results are shown in Table 5.

Equation (29), which relates the deceleration parameter to the dynamical variables generalizes in the matter case to the hyperplane,

$$q = -\frac{\dot{H}}{H^2} - 1 = -(y + K + 1). \quad (43)$$

4.2. Asymptotic analysis

We complete the analysis for the matter case by investigating the asymptotic behavior of the system (41). In order to achieve this we compactify the phase space by transforming to 4-D polar coordinates. The transformation equations are

$$\begin{aligned} D &\rightarrow r \cos \delta, \quad z \rightarrow r \sin \delta \cos \theta, \quad x \rightarrow r \sin \delta \sin \theta \cos \phi, \\ y &\rightarrow r \sin \theta \sin \delta \sin \phi, \end{aligned}$$

where $r \in [0, \infty]$, $\delta \in [0, \pi]$, $\theta \in [0, \pi]$, and $\phi \in [0, 2\pi]$. We then transform the radial coordinate $r \rightarrow \frac{R}{1-R}$ and in the limit $R \rightarrow 1$, the system (40) reduces to

$$\begin{aligned} R' &\rightarrow \cos(\delta)^3 + \cos(\delta) \cos(\theta) \sin(\delta)^2 \sin(\theta) \sin(\phi) + \cos(\delta)^2 \sin(\delta) (\cos(\theta) \\ &\quad + 3 \sin(\theta) \chi) + \sin(\delta)^3 \sin(\theta) \left(\cos(\phi) (2 + \varphi) \right. \\ &\quad \left. + \sin(\phi) (3 \cos(\theta)^2 + 2 \sin(\theta)^2 + \varphi) \right) , \end{aligned} \quad (44)$$

Table 4. Coordinates of the fixed points, the eigenvalues, and solutions for $\exp(-R/\Lambda)$ -gravity in the matter case.

Point	Coordinates(y,z,K,D)	Eigenvalues	Solution
\mathcal{A}_m	$[0, 0, 0, 0]$	$[2 - 3w, 2, 4, 4]$	$a = a_o(t - t_o)^{\frac{1}{2}}$
\mathcal{B}_m	$[0, 0, -1, 0]$	$[2, 2, -2, -(1 + 3w)]$	$a = a_o(t - t_o)$
\mathcal{C}_m	$[0, 0, 0, 3w - 2]$	$[3w - 2, 2, 4, 4]$	$a = a_o(t - t_o)^{\frac{1}{2}}$
\mathcal{D}_m	$[0, 0, -1, 3w + 1]$	$[2, 2, -2, (1 + 3w)]$	$a = a_o(t - t_o)$
\mathcal{E}_m	$[-2, 1, 0, 0]$	$[-\frac{\sqrt{17}+3}{2}, \frac{\sqrt{17}-3}{2}, -2, -3 - 3w]$	$a = a_o e^{\gamma(t-t_o)}$
\mathcal{F}_m	$[-2, 0, 0, 0]$	$[-2, -4, -3w - 4, 0]$	$a = a_o e^{\gamma(t-t_o)}$
\mathcal{G}_m	$[-2, 0, 0, 3w + 4]$	$[3w + 4, -2, -4, 0]$	$a = a_o e^{\gamma(t-t_o)}$

Table 5. Stability of the fixed points for $\exp(-R\Lambda)$ -gravity in the matter case.

Point	$w = 0$	$0 < w < \frac{1}{3}$	$w = \frac{1}{3}$
\mathcal{A}_m	Repeller	Repeller	Repeller
\mathcal{B}_m	Saddle	Saddle	Saddle
\mathcal{C}_m	Saddle	Saddle	Saddle
\mathcal{D}_m	Saddle	Saddle	Saddle
\mathcal{E}_m	Saddle	Saddle	Saddle
\mathcal{F}_m	Saddle-node	Saddle-node	Saddle-node
\mathcal{G}_m	Saddle-node	Saddle-node	Saddle-node
Point	$\frac{1}{3} < w < \frac{2}{3}$	$w = \frac{2}{3}$	$\frac{2}{3} < w < 1$
\mathcal{A}_m	Repeller	Saddle-node	Saddle
\mathcal{B}_m	Saddle	Saddle	Saddle
\mathcal{C}_m	Saddle	Saddle-node	Repeller
\mathcal{D}_m	Saddle	Saddle	Saddle
\mathcal{E}_m	Saddle	Saddle	Saddle
\mathcal{F}_m	Saddle-node	Saddle-node	Saddle-node
\mathcal{G}_m	Saddle-node	Saddle-node	Saddle-node

$$R\delta' \rightarrow \frac{\sin(\delta)\cos(\delta)}{8(R-1)} \left[8\cos(\delta)(\varphi - 1) - \sin(\delta) \left(\cos(3\theta) + 8\cos(\phi)\sin(\theta) + 8\sin(\theta)^3\sin(\phi) + \cos(\theta)(7 + 4\sin(\theta)^2(\cos(2\phi) - \sin(2\phi))) \right) \right], \quad (45)$$

$$R\theta' \rightarrow \frac{\cos(\theta)^2}{2(R-1)} \left[2\cos(\delta)\sin(\phi) + \sin(\delta)(2\cos(\theta)\sin(\phi) + \sin(\theta)(1 - \cos(2\phi) + \sin(2\phi))) \right], \quad (46)$$

Table 6. Coordinates, eigenvalues, and the solutions for fixed points in the asymptotic regime for the $\exp(-R/\Lambda)$ gravity in matter case.

Point	(δ, θ, ϕ)	Eigenvalues	Solution
$\mathcal{A}_m\infty$	$[-\frac{\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}]$	$[1, 0, 0]$	$(N - N_\infty) = [c_1 \pm \frac{3c_\phi}{2}(t - t_o)]^{\frac{2}{3}}$
$\mathcal{B}_m\infty$	$[\frac{\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}]$	$[-1, 0, 0]$	$(N - N_\infty) = [c_1 \pm \frac{c_\phi}{2}(t - t_o)]^2$
$\mathcal{C}_m\infty$	$[-\frac{\pi}{2}, \frac{\pi}{4}, -\frac{\pi}{2}]$	$[-\frac{1}{\sqrt{2}}, 0, 0]$	$(N - N_\infty) = [c_1 \pm \frac{c_\phi}{2}(t - t_o)]^2$
$\mathcal{D}_m\infty$	$[-\frac{3\pi}{4}, \frac{\pi}{2}, -\frac{\pi}{2}]$	$[-\frac{1}{\sqrt{2}}, 0, 0]$	$(N - N_\infty) = [c_1 \pm \frac{c_\phi}{2}(t - t_o)]^2$
$\mathcal{E}_m\infty$	$[\frac{\pi}{2}, \frac{\pi}{4}, -\frac{\pi}{2}]$	$[\frac{1}{\sqrt{2}}, 0, 0]$	$(N - N_\infty) = [c_1 \pm \frac{3c_\phi}{2}(t - t_o)]^{\frac{2}{3}}$
$\mathcal{F}_m\infty$	$[\frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{2}]$	$[\frac{1}{\sqrt{2}}, 0, 0]$	$(N - N_\infty) = [c_1 \pm \frac{3c_\phi}{2}(t - t_o)]^{\frac{2}{3}}$
$\mathcal{M}_m\infty$	$[\frac{\pi}{2}, \frac{\pi}{2}, \phi]$	$[0, 0, \chi]$	
		$y \rightarrow \pm\infty \ K \rightarrow \mp\infty$	$a = a_0 \exp[\gamma(t - t_0)]$
		$y \rightarrow +\infty \ K \rightarrow +\infty$	$(N - N_\infty) = [c_1 \pm \frac{c_\phi}{2}(t - t_o)]^2$
		$y \rightarrow -\infty \ K \rightarrow -\infty$	$(N - N_\infty) = [c_1 \pm \frac{3c_\phi}{2}(t - t_o)]^{\frac{2}{3}}$
$\mathcal{O}_m\infty$	$[\text{arccot}(\chi), -\frac{\pi}{2}, \phi]$	$[0, 0, f(\phi) > 0 \ \forall \phi]$	
		$y \rightarrow \pm\infty \ K \rightarrow \mp\infty$	$a = a_0 \exp[\gamma(t - t_0)]$
		$y \rightarrow +\infty \ K \rightarrow +\infty$	$(N - N_\infty) = [c_1 \pm \frac{c_\phi}{2}(t - t_o)]^2$
		$y \rightarrow -\infty \ K \rightarrow -\infty$	$(N - N_\infty) = [c_1 \pm \frac{3c_\phi}{2}(t - t_o)]^{\frac{2}{3}}$

$$R\phi' \rightarrow \frac{\cos(\phi)(\cos(\delta)\cot(\theta) + \cos(\theta)\sin(\delta)(\sin(\phi) + \cos(\phi) + \cot(\theta)))}{R - 1}, \quad (47)$$

where $\varphi = \cos(\theta)\sin(\theta)\sin(\phi)$. Notice that the first equation of the previous system does not depend on R , which means that the fixed points of this system can be determined by the angular equations alone. The solutions at the fixed points can then be obtained by following the same procedure we used in the vacuum case. All the results are summarized in Table 6. The stability of the first six fixed points are shown in Table 7.

As before, we use the center manifold theorem to determine the stability of these fixed points. In this case, because the eigenvalues have a double zero, the coordinate which correspond to the non-zero eigenvalue are approximated by the function

$$\mathbf{h}(\mathbf{x}_1, \mathbf{x}_2) = a \mathbf{x}_1^2 + b \mathbf{x}_1^2 \mathbf{x}_2^2 + c \mathbf{x}_2^2 + 0(|\mathbf{x}|^3), \quad (48)$$

where \mathbf{x}_1 and \mathbf{x}_2 are the coordinates that correspond to the zero eigenvalues (see the Appendix for details). The stability of the point $\mathcal{M}_m\infty$ depends on the angle ϕ : for $-\frac{\pi}{4} > \phi > \frac{3\pi}{4}$, it is unstable, otherwise it is stable. Finally the point $\mathcal{O}_m\infty$ is unstable for all values of ϕ .

5. Discussion and Conclusions

In this paper we have applied the dynamical system approach to the exponential gravity cosmological model, and found exact solutions together with their stability for both the vacuum and matter cases.

In the vacuum case we found four finite critical points \mathcal{A}_v , \mathcal{B}_v , \mathcal{C}_v and \mathcal{D}_v , of which only two \mathcal{C}_v and \mathcal{D}_v are found to be physical. These last points were found to

Table 7. Stability of the fixed points in non-vacuum $\exp(-R/\Lambda)$ -gravity.

Point	Stability
\mathcal{A}_∞	Stable
\mathcal{B}_∞	Unstable
\mathcal{C}_∞	Unstable
\mathcal{D}_∞	Unstable
\mathcal{E}_∞	Stable
\mathcal{F}_∞	Unstable

represent a solution whose nature depends on the parameter $\gamma(\Lambda)$; for $\Lambda > 0$ we can have either exponential expansion ($\gamma > 0$) or exponential contraction ($\gamma < 0$) and for $\Lambda < 0$ the solution oscillates.

From the stability point of view, the point \mathcal{C}_v , which lies in the invariant submanifold $z = 0$, is of particular interest because, since it is non-hyperbolic, it represent an attractor for $z > 0$ and saddle for $z < 0$, while the other physical point \mathcal{D}_v is found to be a saddle.

On the other hand, the solution connected with the non-physical points \mathcal{A}_v and \mathcal{B}_v are found to correspond to power law evolution and are also interesting because the orbits can approach arbitrarily close to them.

In the asymptotic regime all the critical points represent solutions which have a maximum value for the scale factor, hence all models that evolve to one of the asymptotic future attractors will re-collapse.

The invariant submanifold $z = 0$ divides the phase space into two regions, $z > 0$ and $z < 0$ which correspond to $\Lambda > 0$ and $\Lambda < 0$ respectively. The fact that no orbit can cross the plane $z = 0$ is then consistent with the fact that Λ is a fixed parameter for this model.

In the vacuum case, we found that the region $z < 0$ does not contain any finite critical point. However, in the plane $z = 0$, we have the physical point \mathcal{C}_v which represents a de-Sitter saddle and the non-physical points \mathcal{A}_v and \mathcal{B}_v are a repeller and saddle point respectively. Thus, the only attractors in the region $z < 0$ are asymptotic, which means that all the models that begin their evolution in this region will re-collapse. It is also possible that one can choose initial conditions in such a way that both de-Sitter and power law phases are present in the evolution of the model.

From a physical point of view, the region $z > 0$ appears to be more interesting because the point \mathcal{C}_v represents a de-Sitter attractor for $z > 0$. It follows that there are two different possible solutions towards which models can evolve.

Since the point \mathcal{D}_v , which represents an unstable de-Sitter phase, lies in the region $z > 0$, a set of initial conditions exist for which orbits describe an intermediate de-Sitter phase (see figure 2). Furthermore, for models that evolve near the non-physical point \mathcal{B}_v an intermediate power law phase is also present.

By looking at Figure 3 it is clear that the de-Sitter phases \mathcal{C}_v and \mathcal{D}_v are separated from the past attractor \mathcal{A}_v by the plane $q = 0$, therefore any model that starts near the past attractor \mathcal{A}_v and evolves toward the future de-Sitter attractor \mathcal{C}_v will cross the plane $q = 0$, indicating a transition from an accelerating evolution to a decelerating one.

Table 8. The sectors and the behaviour in each one for the $\exp(-R/\Lambda)$ gravity in matter case. Here $+/-$ corresponds to a positive/negative values of the coordinates.

y	D	z	K	Behaviour	y	D	z	K	Behaviour
+	-	-	-	α	-	-	-	-	β
+	+	-	-	α	-	-	-	+	β
+	-	+	-	α	-	-	+	-	γ
+	-	-	+	α	-	-	+	+	γ
+	+	+	-	α	-	+	+	-	ε
+	-	+	+	α	-	+	+	+	ε
+	+	-	+	α	-	+	-	-	δ
+	+	+	+	α	-	+	-	+	δ

The introduction of matter into this model increases the dimensionality of the phase space, making it more difficult to visualize. On the bases of the relative stability of the fixed points and the invariant submanifold structure it is possible to catalog the possible evolutions of this model in 5 classes (see Table 8).

The first class (α) is characterized by the fact that the cosmic histories evolve towards re-collapse. This class also contains cosmic histories which include an intermediate almost power law transient phase. For the second class (β), cosmic histories evolve towards re-collapse, but in addition to a power law transient phase there can be also an oscillating one. The third class (γ) contains two types of orbits. Depending on the initial conditions, the models will either evolve towards re-collapse or towards a de-Sitter type solution. During this evolution, a transient de Sitter phase can be present. A fourth class (ε) also contains two types of orbits. The universe can either re-collapse or evolve to a de-Sitter type model. In this case there are transient phases that include two different unstable de-Sitter evolutions. The final class (δ) contains models that either re-collapse or end up at an oscillating solution. The possible transient phase in this class are multiple oscillations and/or almost power law behaviour. As it can be seen from Table 8, the condition $D/z > 0$ coming from the weak energy condition, required for standard matter, allows one to exclude completely the classes γ and δ .

In conclusion, $\exp(-\frac{R}{\Lambda})$ gravity has a very rich structure that includes a series of diverse cosmological histories. Particularly important are the ones including multiple de Sitter phases because they could provide us with natural models describing the early and late time acceleration of the Universe. Unfortunately, as is clear from Figure 2, this scenario does not include a decelerated expansion phase between these two de-Sitter phases, so unless some other mechanism is taken into account in these cosmic histories will not admit a standard structure formation scenario.

Appendix A.

In section 3.1 we used the center manifold theorem to analyze the stability of the non-hyperbolic fixed point \mathcal{C}_v in the vacuum case, here we will give a brief review of

this approach [12]. Consider the nonlinear system (bold letters represent vectors)

$$\dot{\mathbf{u}} = f(\mathbf{u}). \quad (\text{A.1})$$

For simplicity we shall assume that the origin is a non-hyperbolic fixed point for this system (this assumption does not affect the generality of our treatment because it is always possible to change the coordinates to make the fixed point the origin of the new coordinate system). If $f \in C^1(E)$ and $f(\mathbf{0}) = \mathbf{0}$, then this system can be written in the diagonal form

$$\dot{\mathbf{u}} = J\mathbf{u} + T(\mathbf{u}), \quad (\text{A.2})$$

where $J = Df(\mathbf{0}) = \text{diag}[Z, P, N]$, the square matrices Z, P, N have r eigenvalues of zero real part, p eigenvalues of positive real part and n eigenvalues of negative real part respectively and $T(\mathbf{u}) = f(\mathbf{u}) - J\mathbf{u}$, where $T \in C^1(E)$, $T(\mathbf{0}) = \mathbf{0}$ and $DT(\mathbf{0}) = \mathbf{0}$. The system (48) can be divided into three subsystems

$$\begin{aligned} \dot{\mathbf{x}} &= Z\mathbf{x} + F(\mathbf{x}, \mathbf{y}, \mathbf{z}), \\ \dot{\mathbf{y}} &= P\mathbf{y} + G(\mathbf{x}, \mathbf{y}, \mathbf{z}), \\ \dot{\mathbf{z}} &= N\mathbf{z} + H(\mathbf{x}, \mathbf{y}, \mathbf{z}), \end{aligned} \quad (\text{A.3})$$

where $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in R^r \times R^p \times R^n$, $\mathbf{F}(\mathbf{0}) = \mathbf{G}(\mathbf{0}) = \mathbf{H}(\mathbf{0}) = \mathbf{0}$, and $D\mathbf{F}(\mathbf{0}) = D\mathbf{G}(\mathbf{0}) = D\mathbf{H}(\mathbf{0}) = \mathbf{0}$. If $(F, G, H) \in C^m(E)$ with $m \geq 1$, it follows from the local center manifold theory that there exist a z -dimensional invariant center manifold $W_{local}^c(\mathbf{0})$ defined by

$$\mathbf{W}_{local}^c(\mathbf{0}) = \{(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in R^r \times R^p \times R^n | \mathbf{y} = \mathbf{h1}(\mathbf{x}), \mathbf{z} = \mathbf{h2}(\mathbf{x}) \text{ for } |\mathbf{x}| < \delta\},$$

for some $\delta > 0$, where $(\mathbf{h1}, \mathbf{h2}) \in C^r(N_\delta(\mathbf{0}))$, $\mathbf{h1}(\mathbf{0}) = \mathbf{h2}(\mathbf{0}) = \mathbf{0}$, $D\mathbf{h1}(\mathbf{0}) = D\mathbf{h2}(\mathbf{0}) = \mathbf{0}$, and they satisfy

$$\begin{aligned} D\mathbf{h1}(\mathbf{x})[Z\mathbf{x} + F(\mathbf{x}, \mathbf{h1}(\mathbf{x}), \mathbf{h2}(\mathbf{x}))] - P\mathbf{h1}(\mathbf{x}) - G(\mathbf{x}, \mathbf{h1}(\mathbf{x}), \mathbf{h2}(\mathbf{x})) &= 0, \\ D\mathbf{h2}(\mathbf{x})[Z\mathbf{x} + F(\mathbf{x}, \mathbf{h1}(\mathbf{x}), \mathbf{h2}(\mathbf{x}))] - N\mathbf{h2}(\mathbf{x}) - H(\mathbf{x}, \mathbf{h1}(\mathbf{x}), \mathbf{h2}(\mathbf{x})) &= 0. \end{aligned}$$

In the neighborhood of a non-hyperbolic fixed point the qualitative behavior of the system (48) is equivalent to the qualitative behavior of the reduced system

$$\dot{\mathbf{x}} = Z\mathbf{x} + F(\mathbf{x}, \mathbf{h1}(\mathbf{x}), \mathbf{h2}(\mathbf{x})). \quad (\text{A.4})$$

The functions $\mathbf{h1}(\mathbf{x}), \mathbf{h2}(\mathbf{x})$ can be approximated by substituting the series expansion of their components into equations (51) and (52). In the case when we have a double zero eigenvalues the coordinate which correspond to the non-zero eigenvalue can be approximated by the function,

$$\mathbf{h}(\mathbf{x}_1, \mathbf{x}_2) = a\mathbf{x}_1^2 + b\mathbf{x}_1^2\mathbf{x}_2^2 + c\mathbf{x}_2^2 + 0(|\mathbf{x}|^3), \quad (\text{A.5})$$

where \mathbf{x}_1 and \mathbf{x}_2 are the coordinates which correspond to the zero eigenvalues. In general the flow on the center manifold near the fixed point takes the form

$$\dot{\mathbf{x}} = a\mathbf{x}^r + \dots, \quad (\text{A.6})$$

If $r \geq 2$ and $a_r \neq 0$, then for r even we have saddle-node at the fixed point, for r odd and $a_r > 0$ we have unstable node and for r odd and $a_r < 0$ we have a topological saddle.

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